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Citation for published version:
Papanicolopulos, S-A & Zervos, A 2012, 'A method for creating a class of triangular C^1 finite elements', *International Journal for Numerical Methods in Engineering*, vol. 89, no. 11, pp. 1437-1450. https://doi.org/10.1002/nme.3296

Digital Object Identifier (DOI):
10.1002/nme.3296

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Early version, also known as pre-print

Published In:
International Journal for Numerical Methods in Engineering

This is the pre-peer reviewed version of the following article: Papanicolopulos, S.-A. and Zervos, A. (2012), A method for creating a class of triangular C^1 finite elements. Int. J. Numer. Meth. Engng., 89: 1437–1450. doi: 10.1002/nme.3296, which has been published in final form at http://onlinelibrary.wiley.com/doi/10.1002/nme.3296/abstract.

A method for creating a class of triangular C^1 finite elements

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SUMMARY

Finite elements providing a C^1 continuous interpolation are useful in the numerical solution of problems where the underlying partial differential equation is of fourth order, such as beam and plate-bending and deformation of strain-gradient-dependent materials. Although a few C^1 elements have been presented in the literature, their development has largely been heuristic, rather than the result of rational design to a predetermined set of desirable element properties. Therefore a general procedure for developing C^1 elements with particular desired properties is still lacking.

This paper presents a methodology by which C^1 elements, such as the TUBA 3 element proposed by Argyris *et al.*, can be constructed. In this method (which, to the best of our knowledge, is the first one of its kind) a class of finite elements is first constructed by requiring a polynomial interpolation and prescribing the geometry, the location of the nodes and the possible types of nodal degrees of freedom. A set of necessary conditions is then imposed to obtain appropriate interpolations. Generic procedures are presented which determine whether a given potential member of the element class meets the necessary conditions. The behaviour of the resulting elements is checked numerically using a benchmark problem in strain-gradient elasticity.

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KEY WORDS: Finite element methods; C^1 element; Triangular element; Gradient elasticity

1. INTRODUCTION

Finite elements employed in problems of solid mechanics usually interpolate only the displacement field. In most cases, the nature of the underlying equations is such that only C^0 continuity is required, that is the interpolation of the displacements must be continuous but not necessarily smooth. In other cases, however, a displacement-only discretisation leads to the requirement for C^1 continuity, that is both the displacements and their derivatives must be continuous. This is the case, for example, in bending of thin plates [1, p. 324], or when strain-gradient models are used [2, 3]. More details on various C^1 elements as well as alternative elements that can be used instead can be found in the book of Zienkiewicz and Taylor [1, pp. 336–376]. In practice, the difficulty in developing appropriate C^1 elements with desirable properties means that mixed formulations are usually preferred [4, 5] or other numerical methods such as boundary elements are employed [6, 7].

A well-known C^1 element is the TUBA 6 triangular element described by Argyris *et al.* [8] (and, independently, by other researchers). This element has 21 degrees of freedom and uses a complete

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Contract/grant sponsor: European Research Council; contract/grant number: ERC 228051

fifth-order polynomial interpolation. A simplified form of this element with 18 degrees of freedom, named TUBA 3, has also been presented. This is often preferred to TUBA 6 due to its simpler implementation, although its interpolation is polynomial but no longer complete. The expressions for the shape functions of TUBA 3 can be found in [9]. For gradient elasticity problems, examples using the TUBA 6 element are given in [10] while examples using the TUBA 3 element are given e.g. in [11, 12].

Due to the simplicity and good numerical behaviour of the TUBA 3 element, it is interesting to investigate the existence of elements with similar properties. Besides other C^1 triangles, this search could extend e.g. to C^1 tetrahedra, or to non-conforming triangles that, while not fully C^1, still provide accurate numerical results. Although Ženíšek [13] provides a theoretical basis for the study of C^1 triangles with polynomial interpolation, the attempt to look for elements similar to TUBA 3 is hindered by the fact that this element is developed as a somehow arbitrary simplification of another, more complex element.

For this reason, in this paper we present for the first time a general method by which C^1 elements such as the TUBA 3 element can be constructed. In this method, a class of finite elements is constructed starting from a formal definition of its characteristics. Generic procedures are then applied that, for a given order of the interpolating polynomial, provide the required interpolation. We apply this method to derive as an example two C^1 elements which are then tested numerically to demonstrate their good numerical behaviour.

2. THE "TRF" ELEMENT CLASS

2.1. Definition of the element class

We consider here a class of elements, called "TRF", defined by the the following characteristics (or "requirements"):

- R1. The elements are triangles with straight edges.
- R2. Only three nodes are used, corresponding to the vertices of the triangle.
- R3. The degrees of freedom at each node are the value of the interpolated function and *all* its derivatives up to order n_d.
- R4. A polynomial interpolation of order n_p is used, which is *not* necessarily complete.
- R5. The interpolation is C^1 continuous.
- R6. The order n_e of *complete* polynomial interpolation must be as high as possible.

We clarify here some of the terms used to describe the polynomial interpolation. The interpolation of a polynomial is "exact" if the interpolation is equal to the polynomial. The "order of interpolation" is the order n_p of the interpolating polynomial. If the interpolation is exact for any polynomial of order n_e but it is not exact for at least one polynomial of order $n_e + 1$, then n_e is the "order of complete interpolation". If $n_e = n_p$, that is if the order of complete interpolation is the same as the order of interpolation, the interpolation is simply called "complete". Note that the interpolation may not be complete even when the interpolating polynomial is complete.

The number of degrees of freedom is $m_d = 3\binom{n_d+2}{2}$, where $\binom{n}{k} = n!/(k!(n-k)!)$, while the number of terms in a complete polynomial of order n_p is $m_p = \binom{n_p+2}{2}$. Since the degrees of freedom must be independent, the relation $m_d \leq m_p$ must hold, resulting in the combinations shown in Table I. As the interpolation is not necessarily complete, we have $1 \leq n_e \leq n_p$. A complete polynomial of order n_e will have $m_e = \binom{n_e+2}{2}$ terms, where $m_e \leq m_d$.

Any particular element of the TRF class presented here is defined by the values n_d, n_p and n_e, and is therefore named TRF$n_d n_p n_e$C1.

2.2. Areal coordinates

Since the element under consideration is a triangle, the formulation can be developed elegantly using areal coordinates instead of Cartesian ones. Referring to Figure 1, the relation between the

Table I. Possible combinations of interpolating polynomial order and order of the derivatives used as degrees of freedom

interpolation order n_p	derivative order n_d	number of terms m_p	number of dofs m_d	max. compl. order n_e
1	0	3	3	1
2	0	6	3	1
3	1	10	9	2
4	1	15	9	2
5	2	21	18	4
6	2	28	18	4
7	3	36	30	6
8	4	45	45	8
9	4	55	45	8
10	5	66	63	9

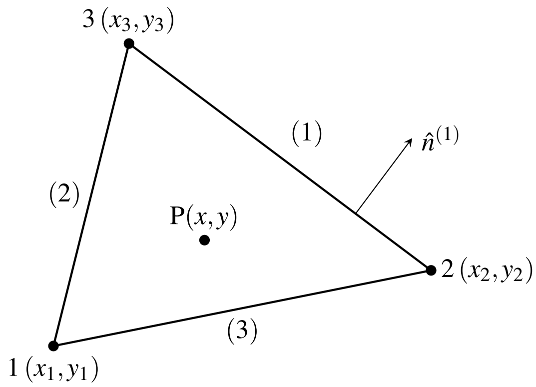

Figure 1. A typical triangular element

Cartesian coordinates x, y of a point P and its areal coordinates L_1, L_2, L_3 is given by

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3 \tag{1}$$

$$y = L_1 y_1 + L_2 y_2 + L_3 y_3 \tag{2}$$

$$1 = L_1 + L_2 + L_3 \tag{3}$$

The relation between Cartesian and areal derivatives is given by

$$\frac{\partial}{\partial x} = \frac{1}{J}\left(b_1 \frac{\partial}{\partial L_1} + b_2 \frac{\partial}{\partial L_2} + b_3 \frac{\partial}{\partial L_3}\right) \tag{4}$$

$$\frac{\partial}{\partial y} = \frac{1}{J}\left(c_1 \frac{\partial}{\partial L_1} + c_2 \frac{\partial}{\partial L_2} + c_3 \frac{\partial}{\partial L_3}\right) \tag{5}$$

where

$$b_1 = y_2 - y_3, \qquad c_1 = x_3 - x_2, \qquad J = b_2 c_3 - c_2 b_3 \tag{6}$$

while similar equations are obtained by cycling the indices $1, 2, 3$. The normal derivative on side 1 is calculated as

$$\frac{\partial}{\partial \hat{\boldsymbol{n}}^{(1)}} = -\frac{1}{\sqrt{b_1^2 + c_1^2}} \frac{1}{J}\left((b_1 b_3 + c_1 c_3)\left(\frac{\partial}{\partial L_3} - \frac{\partial}{\partial L_2}\right) + (b_1^2 + c_1^2)\left(\frac{\partial}{\partial L_1} - \frac{\partial}{\partial L_2}\right)\right) \tag{7}$$

with similar equations obtained for sides 2 and 3 by cycling the indices $1, 2, 3$

In the following we will consider multivariate polynomials of the coordinates, of order n_p. Any such polynomial can be written as

$$v = \boldsymbol{c}^T \boldsymbol{p} \tag{8}$$

where $\boldsymbol{c}$ is a vector of all possible polynomial terms of order up to n_p and $\boldsymbol{p}$ is a vector of polynomial coefficients. Both vectors have $m_p = \binom{n_p+2}{2}$ elements.

If v were expressed in Cartesian coordinates, $\boldsymbol{c}$ would contain all the products of the form $x^\alpha y^\beta$ with $\alpha + \beta \leq n_p$. In areal coordinates, the three coordinates are not independent, therefore $\boldsymbol{c}$ can be written in different ways. We could choose two of the three coordinates, e.g. L_2 and L_3, so that $\boldsymbol{c}$ would similarly only contain all the products $L_2^\alpha L_3^\beta$ with $\alpha + \beta \leq n_p$. In the following we will use however a vector $\boldsymbol{c}$ whose elements are all the products of the form $L_1^\alpha L_2^\beta L_3^\gamma$ with $\alpha + \beta + \gamma = n_p$. This form is equivalent, however it results in simpler expressions, as it does not discriminate against any of the three vertices of the triangle.

2.3. *General form of the interpolation*

The interpolating function w is defined as a linear combination of the degrees of freedom, i.e.

$$w = \boldsymbol{n}^T \boldsymbol{d} \tag{9}$$

where $\boldsymbol{d}$ is the vector of degrees of freedom and $\boldsymbol{n}$ is the vector of shape functions, both with m_d elements. Since we require a polynomial interpolation (requirement R4), the elements of $\boldsymbol{n}$ are themselves multivariate polynomials of total order n_p, therefore using equation (8) we can write

$$\boldsymbol{n} = \boldsymbol{E}^T \boldsymbol{c} \tag{10}$$

where $\boldsymbol{c}$ is the vector of polynomial terms (in the areal coordinates), while $\boldsymbol{E}$ is an $m_p \times m_d$ matrix that depends only on the element geometry. The interpolating polynomial can then be written as

$$w = \boldsymbol{c}^T \boldsymbol{E} \boldsymbol{d} \tag{11}$$

showing clearly the dependence on the coordinates (in $\boldsymbol{c}$), the element geometry (in $\boldsymbol{E}$) and the degrees of freedom $\boldsymbol{d}$.

Out of the requirements given in subsection 2.1, up to now we have only used R4 (while R3 was only used to determine the value of m_d). To obtain the interpolation (9), we need to determine the shape functions $\boldsymbol{n}$, given by equation (10). Since $\boldsymbol{c}$ has already been selected based on the order n_p, the main unknown is the matrix $\boldsymbol{E}$, which will be determined by directly imposing requirements R5 and R6.

2.4. *Imposing C^1 continuity*

The C^1 continuity requirement (R5) is always met within the element due to the polynomial interpolation used. It must however also be satisfied at the common sides between adjacent elements. Since continuity of w along a side i also ensures the continuity of the directional derivative in the direction tangent to the side, we only need to ensure continuity of w and of the normal derivative $\partial w / \partial \hat{\boldsymbol{n}}^{(i)}$.This continuity is obtained if both w and $\partial w / \partial \hat{\boldsymbol{n}}^{(i)}$ on side i depend only on the degrees of freedom belonging to that side and on the geometry of the side (i.e. they do not depend on the degrees of freedom or the coordinates of the third node).

For simplicity, we only consider here continuity across side 1 of the triangle. Continuity across the other two sides will then be obtained by proper cycling of the indices in the resulting expressions. The element has straight edges (requirement R1) therefore on side 1 we have $L_1 = 0$ and $L3 = 1 - L2$. Since only three nodes are used (requirement R2), we can write equation (11) as

$$w = \begin{bmatrix} \boldsymbol{c}_{(Z)} \\ \boldsymbol{c}_{(F)} \\ \boldsymbol{c}_{(H)} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{E}_{(Z1)} & \boldsymbol{E}_{(Z2)} & \boldsymbol{E}_{(Z3)} \\ \boldsymbol{E}_{(F1)} & \boldsymbol{E}_{(F2)} & \boldsymbol{E}_{(F3)} \\ \boldsymbol{E}_{(H1)} & \boldsymbol{E}_{(H2)} & \boldsymbol{E}_{(H3)} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_{(1)} \\ \boldsymbol{d}_{(2)} \\ \boldsymbol{d}_{(3)} \end{bmatrix} \tag{12}$$

where $\boldsymbol{d}_{(i)}$ contains the values of the degrees of freedom of node i, while $\boldsymbol{c}_{(Z)}$, $\boldsymbol{c}_{(F)}$ and $\boldsymbol{c}_{(H)}$ contain the terms of $\boldsymbol{c}$ which are respectively of zero, first or higher order in L_1.

On side 1, $\boldsymbol{c}_{(F)}$ and $\boldsymbol{c}_{(H)}$ are zero, while $\boldsymbol{c}_{(Z)}$ contains the n_p+1 linearly independent products $L_2^{\alpha}(1-L_2)^{n_p-a}$ for $\alpha = 0 \ldots n_p$. The interpolation w on side 1 is therefore given by

$$w|_{(1)} = \boldsymbol{c}_{(Z)}^T \boldsymbol{E}_{(Z1)} \boldsymbol{d}_{(1)} + \boldsymbol{c}_{(Z)}^T \boldsymbol{E}_{(Z2)} \boldsymbol{d}_{(2)} + \boldsymbol{c}_{(Z)}^T \boldsymbol{E}_{(Z3)} \boldsymbol{d}_{(3)} \tag{13}$$

As the degrees of freedom can take arbitrary values and the elements of $\boldsymbol{c}_{(Z)}$ are linearly independent, C^0 continuity is obtained if $\boldsymbol{E}_{(Z1)} = \boldsymbol{0}$ and $\boldsymbol{E}_{(Z2)}$, $\boldsymbol{E}_{(Z3)}$ depend only on b_1 and c_1.

To obtain C^1 continuity, we must also consider the normal derivative of w on side 1. Since the normal derivative of $\boldsymbol{c}_{(H)}$ on side 1 is zero, we obtain

$$\begin{aligned} \frac{\partial w}{\partial \hat{\boldsymbol{n}}^{(1)}} = & \left(\frac{\partial \boldsymbol{c}_{(Z)}}{\partial \hat{\boldsymbol{n}}^{(1)}}\right)^T \left(\boldsymbol{E}_{(Z2)} \boldsymbol{d}_{(2)} + \boldsymbol{E}_{(Z3)} \boldsymbol{d}_{(3)}\right) \\ & + \left(\frac{\partial \boldsymbol{c}_{(F)}}{\partial \hat{\boldsymbol{n}}^{(1)}}\right)^T \left(\boldsymbol{E}_{(F1)} \boldsymbol{d}_{(1)} + \boldsymbol{E}_{(F2)} \boldsymbol{d}_{(2)} + \boldsymbol{E}_{(F3)} \boldsymbol{d}_{(3)}\right) \end{aligned} \tag{14}$$

The elements of the normal derivative of $\boldsymbol{c}_{(F)}$ on side 1 are linearly independent, therefore to obtain C^1 continuity we must set $\boldsymbol{E}_{(F1)} = \boldsymbol{0}$ so that the normal derivative of the interpolation does not depend on the degrees of freedom of node 1. However we note that independence from the geometry (i.e. the position of node 1) is quite more complicated to enforce.

Summarising the above we see that, for the element class presented here and the specific form of the $\boldsymbol{c}$ vector used, the continuity requirement regarding the degrees of freedom is easily enforced by setting to zero the elements of $\boldsymbol{E}$ that belong to a column corresponding to a degree of freedom of node i and to a row corresponding to a term of $\boldsymbol{c}$ where the exponent of L_i is either 0 or 1. We note that the continuity requirement regarding the geometry is more difficult to enforce *a priori*, especially for the normal derivative, and will thus be checked after the order of interpolation is determined by the procedure given in the next subsection.

2.5. *Selecting the order of complete interpolation*

Consider an arbitrary polynomial v of order n_p in the areal coordinates, as given by equation (8). To interpolate v, we must calculate the values of the degrees of freedom. Since these are the values of v and its Cartesian derivatives evaluated at the nodes (requirements R2 and R3), the degrees of freedom $\boldsymbol{d}_v$ can be easily calculated as a linear combination of the coefficients $\boldsymbol{p}$, that is

$$\boldsymbol{d}_v = \boldsymbol{G}\boldsymbol{p} \tag{15}$$

where $\boldsymbol{G}$ is calculated as a function of the element geometry. Indeed, the columns of $\boldsymbol{G}$ are the values of $\boldsymbol{d}$ evaluated for the respective term in $\boldsymbol{c}$. The interpolation w_v of the polynomial v is then given by

$$w_v = \boldsymbol{c}^T \boldsymbol{E}\boldsymbol{G}\boldsymbol{p} \tag{16}$$

where, in the general case, $w_v \neq v$.

Consider now an arbitrary polynomial v_e of order $n_e \leq n_p$, which can be written as

$$v_e = \boldsymbol{c}_e^T \boldsymbol{p}_e \tag{17}$$

where the elements of $\boldsymbol{c}_e$ are all the products of the form $L_1^{\alpha} L_2^{\beta} L_3^{\gamma}$ with $\alpha+\beta+\gamma = n_e$. Since $n_e \leq n_p$, the polynomial terms in $\boldsymbol{c}_e$ can be written as linear combinations of the respective terms in $\boldsymbol{c}$

$$\boldsymbol{c}_e = \boldsymbol{H}_e^T \boldsymbol{c} \tag{18}$$

where $\boldsymbol{H}_e$ is an $m_p \times m_e$ matrix whose elements are known numbers. The polynomial v_e can then be written as

$$v_e = \boldsymbol{c}^T (\boldsymbol{H}_e \boldsymbol{p}_e) \tag{19}$$

therefore its interpolation w_e is given by equation (16) as

$$w_e = \boldsymbol{c}^T \boldsymbol{E}\boldsymbol{G}\boldsymbol{H}_e \boldsymbol{p}_e \tag{20}$$

Imposing requirement R6, i.e. requiring that $w_v = v_e$, and taking into account that the elements of $\boldsymbol{c}$ and $\boldsymbol{p}_e$ can take any value, equations (19) and (20) yield

$$(\boldsymbol{E}\boldsymbol{G} - \boldsymbol{I})\boldsymbol{H}_e = \boldsymbol{0} \tag{21}$$

Using this equation, the largest possible value of n_e which yields a solution for $\boldsymbol{E}$ is determined by trial and error, noting that only $\boldsymbol{H}_e$ depends on n_e.

The above procedure requires the evaluation of the different $\boldsymbol{H}_e$ matrices for different values of n_e. We give here a slightly less obvious method, which only requires evaluating a single additional matrix. Following a comment made in section 2.2, we introduce a vector $\tilde{\boldsymbol{c}}$ whose elements are all the products $L_2^\alpha L_3^\beta$ with $\alpha + \beta \leq n_p$, which are *sorted* in ascending total polynomial order. The relation between $\tilde{\boldsymbol{c}}$ and $\boldsymbol{c}$ is given by

$$\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{H}}^T \boldsymbol{c} \tag{22}$$

where $\tilde{\boldsymbol{H}}$ is an $m_p \times m_p$ matrix whose elements are all constant numbers. The advantage of this method is that the respective vector $\tilde{\boldsymbol{c}}_e$ for the polynomial of order n_e consists of the first m_e elements of $\tilde{\boldsymbol{c}}$, so we can write

$$\tilde{\boldsymbol{c}}_e = \boldsymbol{I}_e^T \tilde{\boldsymbol{c}} \tag{23}$$

where $\boldsymbol{I}_e$ is an $m_p \times m_e$ matrix with diagonal elements equal to one and off-diagonal elements equal to zero. After some calculations we get, instead of equation (21), the equation

$$(\boldsymbol{E}\boldsymbol{G} - \boldsymbol{I})\tilde{\boldsymbol{H}}\boldsymbol{I}_e = \boldsymbol{0} \tag{24}$$

In this form, n_e only affects $\boldsymbol{I}_e$, by determining that only m_e of the columns of $(\boldsymbol{E}\boldsymbol{G} - \boldsymbol{I})\tilde{\boldsymbol{H}}$ should be equated to zero.

2.6. *Interpolation of the interpolation*

Equation (16) gives the interpolation w_v of a polynomial v for which, as already mentioned, in the general case $w_v \neq v$. Since w_v is itself a polynomial, its interpolation w_w is given by

$$w_w = \boldsymbol{c}^T \boldsymbol{E}\boldsymbol{G}\boldsymbol{E}\boldsymbol{G}\boldsymbol{p} \tag{25}$$

where in the general case $w_w \neq w_v$. We may however impose the requirement $w_w = w_v$, in which case equations (16) and (25) yield

$$\boldsymbol{E}\boldsymbol{G}\boldsymbol{E}\boldsymbol{G} = \boldsymbol{E}\boldsymbol{G} \tag{26}$$

which, if $\boldsymbol{G}\boldsymbol{E}$ is invertible, yields

$$\boldsymbol{G}\boldsymbol{E} = \boldsymbol{I} \tag{27}$$

It should be emphasised again that it is not necessary to require that the interpolation of the interpolating polynomial must be exact. As shown when considering specific elements, however, if this requirement can be imposed then using equation (27) we can easily evaluate most of the elements of $\boldsymbol{E}$.

3. THE TRF254C1 ELEMENT

To demonstrate the validity of the method we presented, we apply it here for the case of an element that uses a fifth-order interpolation, where we expect to derive the TUBA 3 element.

3.1. Element formulation

For an element that uses a fifth-order interpolation, like TUBA 3, we have $n_p = 5$ and $m_p = 21$. The elements of vector $\boldsymbol{c}$ are therefore

$$
\begin{aligned}
\boldsymbol{c} = [&L_1^5, L_1^4L_2, L_1^4L_3, L_1^3L_2^2, L_1^3L_3^2, L_1^3L_2L_3,\\
&L_2^5, L_2^4L_3, L_2^4L_1, L_2^3L_3^2, L_2^3L_1^2, L_2^3L_3L_1,\\
&L_3^5, L_3^4L_1, L_3^4L_2, L_3^3L_1^2, L_3^3L_2^2, L_3^3L_1L_2,\\
&L_1L_2^2L_3^2, L_2L_3^2L_1^2, L_3L_1^2L_2^2]
\end{aligned}
\tag{28}
$$

In this case, as easily seen from Table I, $n_d = 2$ and $m_d = 18$ so that the degrees of freedom for the interpolation of a function f are

$$
\begin{aligned}
\boldsymbol{d} = [&f^{(1)}, f^{(1)}_{,x}, f^{(1)}_{,y}, f^{(1)}_{,xx}, f^{(1)}_{,yy}, f^{(1)}_{,xy},\\
&f^{(2)}, f^{(2)}_{,x}, f^{(2)}_{,y}, f^{(2)}_{,xx}, f^{(2)}_{,yy}, f^{(2)}_{,xy},\\
&f^{(3)}, f^{(3)}_{,x}, f^{(3)}_{,y}, f^{(3)}_{,xx}, f^{(3)}_{,yy}, f^{(3)}_{,xy}]
\end{aligned}
\tag{29}
$$

where the superscript in parentheses indicates the node where f and its derivatives are evaluated while the subscripts after the comma indicate differentiation with respect to the Cartesian coordinates.

Due to the order of the elements in vectors $\boldsymbol{c}$ and $\boldsymbol{d}$, the $\boldsymbol{G}$ matrix has the convenient form

$$
\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_1 & \cdot & \cdot & \cdot \\ \cdot & \boldsymbol{G}_2 & \cdot & \cdot \\ \cdot & \cdot & \boldsymbol{G}_3 & \cdot \end{bmatrix}
\tag{30}
$$

where $\boldsymbol{G}_j$ are 6×6 matrices that depend only on the quantities b_i and c_i ($i = 1, 2, 3$), and can be determined from each other through cyclic permutation of the indices.

Equation (21) together with the restrictions for C^1 continuity can be satisfied for order of complete interpolation $n_e = 4$, therefore the element under consideration is a TRF254C1 element. After extensive calculations we obtain

$$
\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_1 & \cdot & \cdot \\ \cdot & \boldsymbol{E}_2 & \cdot \\ \cdot & \cdot & \boldsymbol{E}_3 \\ \boldsymbol{E}_1^* & \boldsymbol{E}_2^* & \boldsymbol{E}_3^* \end{bmatrix}
\tag{31}
$$

where

$$
\boldsymbol{E}_1 = \begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
5 & c_3 & -b_3 & \cdot & \cdot & \cdot \\
5 & -c_2 & b_2 & \cdot & \cdot & \cdot \\
10 & 4c_3 & -4b_3 & \frac{1}{2}c_3^2 & \frac{1}{2}b_3^2 & -b_3c_3 \\
10 & -4c_2 & 4b_2 & \frac{1}{2}c_2^2 & \frac{1}{2}b_2^2 & -b_2c_2 \\
20 & 4c_3 - 4c_2 & 4b_2 - 4b_3 & -c_3c_2 & -b_3b_2 & b_2c_3 + c_2b_3
\end{bmatrix}
\tag{32}
$$

and

$$
\left(\boldsymbol{E}_1^*\right)^T = \begin{bmatrix}
0 & 30r_{21} & 30r_{31} \\
0 & -(3c_1 + 15c_2r_{21}) & 3c_1 + 15c_3r_{31} \\
0 & 3b_1 + 15b_2r_{21} & -(3b_1 + 15b_3r_{31}) \\
0 & c_2c_1 + \frac{5}{2}c_2^2r_{21} & c_3c_1 + \frac{5}{2}c_3^2r_{31} \\
0 & b_2b_1 + \frac{5}{2}b_2^2r_{21} & b_3b_1 + \frac{5}{2}b_3^2r_{31} \\
0 & -(b_1c_2 + b_2c_1 + 5b_2c_2r_{21}) & -(b_1c_3 + b_3c_1 + 5b_3c_3r_{31})
\end{bmatrix}
\tag{33}
$$

The submatrices $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are obtained from $\boldsymbol{E}_1$ through cyclic permutation of the indices while the submatrices $\boldsymbol{E}_2^*$ and $\boldsymbol{E}_3^*$ are obtained from $\boldsymbol{E}_1^*$ through cyclic permutation of the indices and cyclic permutation of their rows (shown in equation (33) as the columns).

The quantities r_{ij} in equation (33) are defined as

$$r_{ij} = \frac{b_i c_j - b_j c_i}{b_i^2 + c_i^2}\alpha(b_i, c_i) - \frac{b_i b_j + c_i c_j}{b_i^2 + c_i^2} \tag{34}$$

where $\alpha(b_i, c_i)$ is a dimensionless function introduced when imposing C^1 continuity with respect to the geometry, for which

$$\alpha(b_i, c_i) = \alpha(-b_i, -c_i) \tag{35}$$

The presence of the function $\alpha(b_i, c_i)$ in the expression for $\boldsymbol{E}$ shows that there is not a unique TRF254C1 element, but rather a family of them. For $\alpha(b_i, c_i) = 0$ we obtain the simplest expression for $\boldsymbol{E}$ and it is easy to verify that the resulting element is the TUBA 3 element.

Since the requirements of section 2.1 are met for any admissible choice of $\alpha(b_i, c_i)$, we expect the resulting elements to have equally good numerical behaviour, and this is indeed found to be the case in the numerical results obtained in section 6. The choice $\alpha(b_i, c_i) = 0$ is however preferable since it allows for simpler application of boundary conditions, as shown in section 3.2.

Note that the values of $\boldsymbol{E}$ and $\boldsymbol{G}$ obtained for the element TRF254C1 satisfy equation (27), therefore the interpolation of the interpolation is exact. Indeed, we could have used equation (27) from the beginning, to directly calculate the submatrices $\boldsymbol{E}_i$ as

$$\boldsymbol{E}_i = \boldsymbol{G}_i^{-1} \qquad \text{for } i = 1, 2, 3 \tag{36}$$

3.2. *Interpolation on the boundary*

We consider, for simplicity, the case of an element whose side 1 is parallel to the x axis in the Cartesian space, so that $b_1 = 0$. The interpolation $w_{(1)}$ of a function f on side 1 is then calculated as

$$\begin{aligned} w_{(1)} = {} & L_2^3(6L_2^2 - 15L_2 + 10)f^{(2)} - L_2^3(1 - L_2)(3L_2 - 4)c_1 f^{(2)}_{,x} + \tfrac{1}{2}L_2^3(1 - L_2)^2 c_1^2 f^{(2)}_{,xx} \\ & + (1 - L_2)^3(6L_2^2 + 3L_2 + 1)f^{(3)} - (1 - L_2)^3 L_2(3L_2 + 1)c_1 f^{(3)}_{,x} + \tfrac{1}{2}(1 - L_2)^3 L_2^2 c_1^2 f^{(3)}_{,xx} \end{aligned} \tag{37}$$

We see that the interpolation on the side (that is, along the x direction) only depends on the value of f and its derivatives with respect to x (evaluated on nodes 2 and 3). The value of the normal derivative on side 1 is similarly calculated as

$$\begin{aligned} |c_1|\frac{\partial w}{\partial \hat{\boldsymbol{n}}^{(1)}} = {} & c_1 L_2^2(2L_2 - 3)f^{(2)}_{,y} - c_1^2 L_2^2(1 - L_2)f^{(2)}_{,xy} - c_1(1 - L_2)^2(1 + 2L_2)f^{(3)}_{,y} \\ & + c_1^2(1 - L_2)^2 L_2 f^{(3)}_{,xy} + \alpha_1 L_2^2(1 - L_2)^2\Big(-f^{(2)} - (1/2)c_1 f^{(2)}_{,x} \\ & - (1/12)c_1^2 f^{(2)}_{,xx} + f^{(3)} - (1/2)c_1 f^{(3)}_{,x} + (1/12)c_1^2 f^{(3)}_{,xx}\Big) \end{aligned} \tag{38}$$

Therefore, setting $\alpha_1 = 0$ (and similarly $\alpha_2 = \alpha_3 = 0$) the value of the interpolation and its normal derivative on the boundary can be prescribed independently.

4. THE TRF375C1 ELEMENT

Using a seventh-order interpolation, i.e. $n_p = 7$ and $m_p = 36$, the vector $\boldsymbol{c}$ can be written as

$$\begin{aligned} \boldsymbol{c} = [& L_1^7, L_1^6 L_2, L_1^6 L_3, L_1^5 L_2^2, L_1^5 L_3^2, L_1^5 L_2 L_3, L_1^4 L_2^3, L_1^4 L_3^3, L_1^4 L_2^2 L_3, L_1^4 L_3^2 L_2, \\ & L_2^7, L_2^6 L_3, L_2^6 L_1, L_2^5 L_3^2, L_2^5 L_1^2, L_2^5 L_3 L_1, L_2^4 L_3^3, L_2^4 L_1^3, L_2^4 L_3^2 L_1, L_2^4 L_1^2 L_3, \\ & L_3^7, L_3^6 L_1, L_3^6 L_2, L_3^5 L_1^2, L_3^5 L_2^2, L_3^5 L_1 L_2, L_3^4 L_1^3, L_3^4 L_2^3, L_3^4 L_1^2 L_2, L_3^4 L_2^2 L_1, \\ & L_2^3 L_3^3 L_1, L_3^3 L_1^3 L_2, L_1^3 L_2^3 L_3, L_1^3 L_2^2 L_3^2, L_2^3 L_3^2 L_1^2, L_3^3 L_1^2 L_2^2] \end{aligned} \tag{39}$$

In this case $n_d = 3$ and $m_d = 30$, so the vector $\boldsymbol{d}$ of degrees of freedom is

$$\begin{aligned}\boldsymbol{d} = [&f^{(1)}, f^{(1)}_{,x}, f^{(1)}_{,y}, f^{(1)}_{,xx}, f^{(1)}_{,yy}, f^{(1)}_{,xy}, f^{(1)}_{,xxx}, f^{(1)}_{,yyy}, f^{(1)}_{,xxy}, f^{(1)}_{,yyx},\\ &f^{(2)}, f^{(2)}_{,x}, f^{(2)}_{,y}, f^{(2)}_{,xx}, f^{(2)}_{,yy}, f^{(2)}_{,xy}, f^{(2)}_{,xxx}, f^{(2)}_{,yyy}, f^{(2)}_{,xxy}, f^{(2)}_{,yyx},\\ &f^{(3)}, f^{(3)}_{,x}, f^{(3)}_{,y}, f^{(3)}_{,xx}, f^{(3)}_{,yy}, f^{(3)}_{,xy}, f^{(3)}_{,xxx}, f^{(3)}_{,yyy}, f^{(3)}_{,xxy}, f^{(3)}_{,yyx}]\end{aligned} \tag{40}$$

Interestingly, the $\boldsymbol{G}$ matrix has the same form as the one for the TRF254C1 element, given in equation (30), where G_j are now 10×10 matrices.

Choosing $n_e = 6$ provides no C^1 element, while $n_e = 5$ produces a TRF375C1 element. As in the case of the TRF254C1 element, a family of elements is actually obtained, which now depend on 24 functions. Of these, 12 functions can be determined using equation (27), which was not automatically satisfied by imposing C^1 continuity. The $\boldsymbol{E}$ matrix then takes the form

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_1 & \cdot & \cdot \\ \cdot & \boldsymbol{E}_2 & \cdot \\ \cdot & \cdot & \boldsymbol{E}_3 \\ \boldsymbol{E}_1^* & \boldsymbol{E}_2^* & \boldsymbol{E}_3^* \\ \boldsymbol{E}_1^{**} & \boldsymbol{E}_2^{**} & \boldsymbol{E}_3^{**} \end{bmatrix} \tag{41}$$

where, as in the TRF254C1 element, $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are obtained from $\boldsymbol{E}_1 = \boldsymbol{G}_1^{-1}$ by cyclic permutation of the indices while the starred submatrices are obtained from each other by cyclic permutation of the indices and of the rows.

The 3×10 matrices $\boldsymbol{E}_i^*$ $(i = 1, 2, 3)$ have three functions which can be determined by requiring, as in section 3.2, that on the boundary the value of the interpolation and its normal derivative can be prescribed independently.

There remain therefore 9 functions in the 3×10 matrices $\boldsymbol{E}_i^{**}$ $(i = 1, 2, 3)$ which cannot be determined by C^1 requirements. This happens because these matrices multiply the terms $L_1^3 L_2^2 L_3^2$, $L_2^3 L_3^2 L_1^2$ and $L_3^3 L_1^2 L_2^2$ respectively. These terms are "C^1 bubble functions" since their value and the value of their normal derivative on the boundary is zero, therefore the components of $\boldsymbol{E}_i^{**}$ are only determined by requiring $n_e = 5$. The value of the 9 functions is then selected arbitrarily, with the objective of providing relatively simple expressions.

The resulting expressions for the submatrices of the $\boldsymbol{E}$ matrix are quite large. It is therefore simpler to give directly the expressions for the first ten shape functions (with the two remaining sets of ten shape functions being obtained by cyclic rotation of the indices). These are

$$\begin{aligned}
n_1 = {} & L_1^4(-20L_1^3 + 70L_1^2 - 84L_1 + 35) + 30L_1^2 L_2^2 L_3^2(4L_1 + 1)\\
& + 140L_3^3 L_1^3 L_2 r_{21} + 140L_1^3 L_2^3 L_3 r_{31}\\
n_2 = {} & L_1^2 L_2(30L_1 L_3^2 L_2 + 9L_2 L_3^2 - 10L_1 L_3 L_2^2 + 10L_3^3 L_1 + 15L_1^2 - 24L_1^3 + 10L_1^4)c_3\\
& - L_1^2 L_3(30L_1 L_3 L_2^2 + 9L_2^2 L_3 - 10L_1 L_3^2 L_2 + 10L_1 L_2^3 + 15L_1^2 - 24L_1^3 + 10L_1^4)c_2\\
& - 70L_3^3 L_1^3 L_2 r_{21} c_2 + 70L_1^3 L_2^3 L_3 r_{31} c_3\\
n_3 = {} & -L_1^2 L_2(30L_1 L_3^2 L_2 + 9L_2 L_3^2 - 10L_1 L_3 L_2^2 + 10L_3^3 L_1 + 15L_1^2 - 24L_1^3 + 10L_1^4)b_3\\
& + L_1^2 L_3(30L_1 L_3 L_2^2 + 9L_2^2 L_3 - 10L_1 L_3^2 L_2 + 10L_1 L_2^3 + 15L_1^2 - 24L_1^3 + 10L_1^4)b_2\\
& + 70L_3^3 L_1^3 L_2 r_{21} b_2 - 70L_1^3 L_2^3 L_3 r_{31} b_3\\
n_4 = {} & (1/2)L_1^2 L_2^2(4L_1 L_3^2 + (3/2)L_3^2 - 4L_1^3 + 5L_1^2 - 8L_3 L_2 L_1)c_3^2\\
& + (1/2)L_3^2 L_1^2(4L_1 L_2^2 + (3/2)L_2^2 - 4L_1^3 + 5L_1^2 - 8L_3 L_2 L_1)c_2^2\\
& - L_2 L_1^2 L_3(-3L_1^2 + 4L_1 + 3L_2 L_3)c_2 c_3 + 14L_3^3 L_1^3 L_2 r_{21} c_2^2 + 14L_1^3 L_2^3 L_3 r_{31} c_3^2\\
n_5 = {} & (1/2)L_1^2 L_2^2(4L_1 L_3^2 + (3/2)L_3^2 - 4L_1^3 + 5L_1^2 - 8L_3 L_2 L_1)b_3^2\\
& + (1/2)L_3^2 L_1^2(4L_1 L_2^2 + (3/2)L_2^2 - 4L_1^3 + 5L_1^2 - 8L_3 L_2 L_1)b_2^2\\
& - L_2 L_1^2 L_3(-3L_1^2 + 4L_1 + 3L_2 L_3)b_2 b_3 + 14L_3^3 L_1^3 L_2 r_{21} b_2^2 + 14L_1^3 L_2^3 L_3 r_{31} b_3^2
\end{aligned}$$

$$\begin{aligned}
n_6 = &-L_1^2L_2^2(4L_1L_3^2+(3/2)L_3^2-4L_1^3+5L_1^2-8L_3L_2L_1)c_3b_3\\
&-L_3^2L_1^2(4L_1L_2^2+(3/2)L_2^2-4L_1^3+5L_1^2-8L_3L_2L_1)b_2c_2\\
&+L_2L_1^2L_3(-3L_1^2+4L_1+3L_2L_3)(b_2c_3+c_2b_3)\\
&-28L_3^3L_1^3L_2r_{21}b_2c_2-28L_1^3L_2^3L_3r_{31}c_3b_3\\
n_7 = &\quad(1/6)L_1^3L_2^3(L_1-3L_3)c_3^3-(1/6)L_1^3L_3^3(L_1-3L_2)c_2^3\\
&-(1/4)L_1^2L_2^2L_3(2L_1+L_3)c_3^2c_2+(1/4)L_1^2L_3^2L_2(2L_1+L_2)c_2^2c_3\\
&+(7/6)L_1^3L_2^3L_3r_{31}c_3^3-(7/6)L_1^3L_3^3L_2r_{21}c_2^3\\
n_8 = &-(1/6)L_1^3L_2^3(L_1-3L_3)b_3^3+(1/6)L_1^3L_3^3(L_1-3L_2)b_2^3\\
&+(1/4)L_1^2L_2^2L_3(2L_1+L_3)b_3^2b_2-(1/4)L_1^2L_3^2L_2(2L_1+L_2)b_2^2b_3\\
&-(7/6)L_1^3L_2^3L_3r_{31}b_3^3+(7/6)L_1^3L_3^3L_2r_{21}b_2^3\\
n_9 = &-(1/2)L_1^3L_2^3(L_1-3L_3)c_3^2b_3+(1/2)L_1^3L_3^3(L_1-3L_2)c_2^2b_2\\
&+(1/4)L_1^2L_2^2L_3(2L_1+L_3)(2c_3c_2b_3+c_3^2b_2)\\
&-(1/4)L_1^2L_3^2L_2(2L_1+L_2)(2b_2c_2c_3+b_3c_2^2)\\
&-(7/2)L_1^3L_2^3L_3r_{31}c_3^2b_3+(7/2)L_3^3L_1^3L_2r_{21}c_2^2b_2\\
n_{10} = &\quad(1/2)L_1^3L_2^3(L_1-3L_3)b_3^2c_3-(1/2)L_1^3L_3^3(L_1-3L_2)b_2^2c_2\\
&-(1/4)L_1^2L_2^2L_3(2L_1+L_3)(2b_3b_2c_3+b_3^2c_2)\\
&+(1/4)L_1^2L_3^2L_2(2L_1+L_2)(2b_2c_2b_3+b_2^2c_3)\\
&+(7/2)L_1^3L_2^3L_3r_{31}b_3^2c_3-(7/2)L_3^3L_1^3L_2r_{21}b_2^2c_2
\end{aligned}$$

where the quantities r_{ij} are obtained from equation (34) for $\alpha(b_i, c_i) = 0$.

Finally, it is worth noting that the element thus obtained is not a direct simplification of the C^1 full heptic triangle (see e.g. [14]), since the latter does not have third-order derivatives as degrees of freedom but it has internal nodes (corresponding to the three "C^1 bubble functions" of the seventh-order polynomial).

5. OTHER ELEMENTS IN THE TRF CLASS

The TRF254C1 element obtained in section 3 for $n_p = 5$ is the first element in the TRF class, i.e. the one with the smaller number of degrees of freedom. Although this follows from the proof of Ženíšek [13], the development of lower-order interpolation elements was attempted, verifying that in these cases the procedure presented here indeed shows that such elements do not exist.

For interpolation order $n_p = 6$ the best possible element would be a TRF264C1 element that would not offer any advantage in terms of quality of interpolation over the TRF254C1 element, therefore the next useful element is the TRF375C1 element described in section 4.

It is worth considering briefly the case $n_p = 8$ for which table I implies that we may obtain a TRF488C1 element which, in contrast to the elements considered up to now, would have the useful characteristic of using a complete polynomial interpolation. In this case, with $n_e = n_p$, equation (24) yields

$$\boldsymbol{E}\boldsymbol{G} = \boldsymbol{I} \tag{42}$$

since $\boldsymbol{I}_e$ is now simply the identity matrix $\boldsymbol{I}$ and $\tilde{\boldsymbol{H}}$ is invertible. In other words, if $n_e = n_p$ then $\boldsymbol{E}$ is just the inverse of $\boldsymbol{G}$ (since both matrices are now square). However for $n_p = 8$ and $n_d = 4$ the $\boldsymbol{G}$ matrix is singular, since the columns corresponding to the terms $L_1^2L_2^3L_3^3$, $L_2^2L_3^3L_1^3$ and $L_3^2L_1^3L_2^3$ are all zero, and therefore a TRF488C1 element does not exist.

Using the procedure described in this paper, additional elements can be created for higher interpolation orders, for example a TRF497C1 element, although in usual finite element applications they would probably see little use.

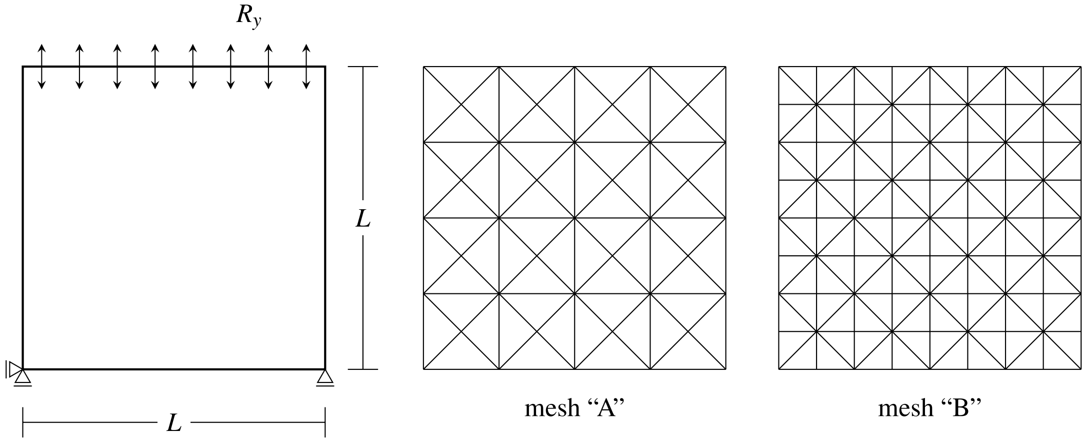

Figure 2. Benchmark problem of a square domain with an applied double traction: Geometry, loads and types of mesh used.

6. NUMERICAL EXAMPLES

We present in this section a brief evaluation of the numerical behaviour of the elements described in this paper, using a benchmark problem in gradient elasticity [15, 16]. The problem, shown in Figure 2, can be considered as a simplified version of the problem of the “bolted layer” proposed by Vardoulakis [17].

The constitutive model used is the commonly encountered “simplified” linear isotropic gradient elasticity model, with a single internal length ℓ (see for example [18]). The material parameters used in the numerical solution are $\lambda = 0$, $\mu = 500$ and $\ell = 0.005$, the dimension of the domain is $L = 0.1$ and the applied double traction is $R_y = 1$. The value $\lambda = 0$ is selected to provide a simple but non-trivial analytical solution without the need for special measures at the left and right boundaries. This allows a more accurate evaluation of the numerical behaviour of each element type used. Two different types of meshes are used, neither of which presents a single “preferred” direction.

As a measure of the accuracy of the numerical solution we compute the relative error of the vertical displacement u_y^{t} at the top edge. The relative error is defined as the absolute value of the difference between analytical and numerical solution, normalised by the analytical solution (the analytical solution in this case is $u_y^{\mathrm{t}} = R_y/2\mu$). The results, for the TRF254C1 element with different constant values α of the $\alpha(b_i, c_i)$ function as well as for the TRF375C1 element, are plotted as convergence diagrams in Figure 3 showing the relative error as a function of the mesh density, for each of the two meshes.

Considering the TRF254C1 element, we see that for all values of α the numerical solution converges to the analytical one, up to a point where round-off errors become important. Negative values of α were also tested with results very similar to those for the corresponding positive values. As expected, for larger values of the α parameter round-off errors affect the numerical solution for smaller problem sizes. Comparison of the results obtained with the two different mesh types shows that the choice $\alpha = 0$ provides good accuracy which, although not always the best possible, it may nevertheless be trusted to consistently improve upon mesh refinement. Since there does not seem to exist a specific choice of α that consistently yields better results, the value $\alpha = 0$ is preferred for the reasons given in Section 3.2.

The TRF375C1 element yields better accuracy than the TRF254C1 for both meshes considered here, as well as a slightly higher convergence rate. For the case of domains with curved boundaries, however, the TRF254C1 element may be preferrable as for the same total number of degrees of freedom, more elements (with straight edges) would be used thus allowing better approximation of the geometry.

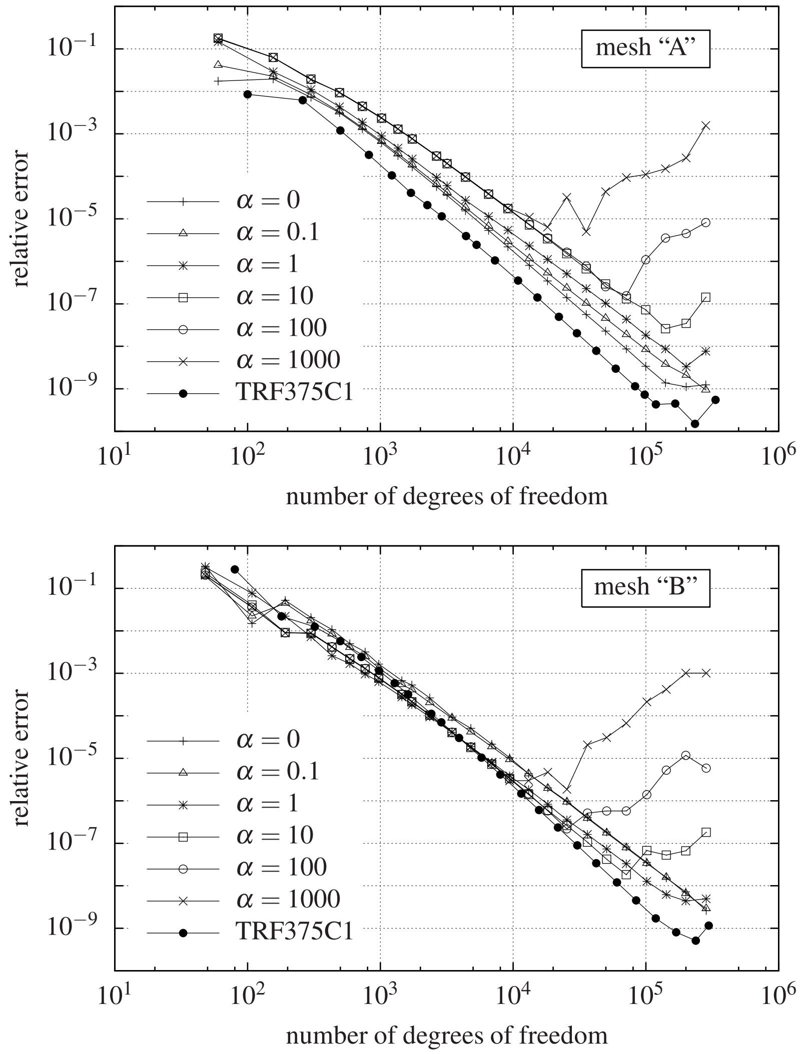

Figure 3. Convergence diagram for the benchmark problem for each mesh type.

7. CONCLUSIONS

While the TUBA 3 triangle is frequently used in applications requiring C^1 elements, it is always considered as a simplified version of the more complex TUBA 6 element. In this paper we describe for the first time a generic procedure through which a class of elements like TUBA 3 can be directly generated, without reference to other, more complex elements.

Application of this procedure shows how the TUBA 3 element is not the only member of this class providing the given interpolation order, but it is the simpler one to formulate and, more importantly, to use when enforcing boundary conditions. The same procedure can be used to derive a TRF375C1 element that includes third-order derivatives as degrees of freedom. Numerical solutions to a problem of gradient elasticity are provided to validate the theoretical results obtained.

The proposed procedure refers to a specific class of elements. It however provides a systematic framework which could be further extended to investigate C^1 triangular elements with other types of degrees of freedom, non-conforming triangular elements, as well as C^1 tetrahedral elements. Additionally, the method can be extended without great difficulty to consider elements with higher continuity, for example C^2 elements.

ACKNOWLEDGEMENTS

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007–2013) / ERC grant agreement n^{o} 228051 [MEDIGRA]. A. Zervos also acknowledges the support of the Faculty of Engineering and the Environment, University of Southampton.

REFERENCES

1. Zienkiewicz O, Taylor R. *The Finite Element Method for Solid and Structural Mechanics*. 6^{th} edn., Elsevier Butterworth-Heinemann, 2005.
2. Zervos A, Papanastasiou P, Vardoulakis I. A finite element displacement formulation for gradient elastoplasticity. *International Journal for Numerical Methods in Engineering* 2001; **50**(6):1369–1388, doi:10.1002/1097-0207(20010228)50:6<1369::AID-NME72>3.0.CO;2-K.
3. Papanicolopulos SA, Zervos A, Vardoulakis I. A three dimensional C^1 finite element for gradient elasticity. *International Journal for Numerical Methods in Engineering* 2009; **77**(10):1396–1415, doi:10.1002/nme.2449.
4. Matsushima T, Chambon R, Caillerie D. Large strain finite element analysis of a local second gradient model: application to localization. *International Journal for Numerical Methods in Engineering* 2002; **54**(4):499–521, doi: 10.1002/nme.433.
5. Askes H, Gutiérrez MA. Implicit gradient elasticity. *International Journal for Numerical Methods in Engineering* 2006; **67**(3):400–416, doi:10.1002/nme.1640.
6. Karlis GF, Tsinopoulos SV, Polyzos D, Beskos DE. Boundary element analysis of mode I and mixed mode (I and II) crack problems of 2-D gradient elasticity. *Computer Methods in Applied Mechanics and Engineering* 2007; **196**(49–52):5092–5103, doi:10.1016/j.cma.2007.07.006.
7. Karlis GF, Charalambopoulos A, Polyzos D. An advanced boundary element method for solving 2D and 3D static problems in Mindlin's strain-gradient theory of elasticity. *International Journal for Numerical Methods in Engineering* 2010; **83**(11):1407–1427, doi:10.1002/nme.2862.
8. Argyris JH, Fried I, Scharpf DW. The TUBA family of plate elements for the matrix displacement method. *Aeronaut. J. R. Aeronaut. Soc.* 1968; **72**(692):701–709.
9. Dasgupta S, Sengupta D. A higher-order triangular plate bending element revisited. *International Journal for Numerical Methods in Engineering* 1990; **30**:419–430, doi:10.1002/nme.1620300303.
10. Fischer P, Mergheim J, Steinmann P. On the C^1 continuous discretization of non-linear gradient elasticity: A comparison of NEM and FEM based on Bernstein-Bézier patches. *International Journal for Numerical Methods in Engineering* 2010; **82**(10):1282–1307, doi:10.1002/nme.2802.
11. Zervos A, Papanicolopulos SA, Vardoulakis I. Two finite element discretizations for gradient elasticity. *Journal of Engineering Mechanics–ASCE* 2009; **135**(3):203–213, doi:10.1061/(ASCE)0733-9399(2009)135:3(203).
12. Akarapu S, Zbib HM. Numerical analysis of plane cracks in strain-gradient elastic materials. *International Journal of Fracture* 2006; **141**(3–4):403–430, doi:10.1007/s10704-006-9004-y.
13. Ženíšek A. Interpolation polynomials on the triangle. *Numerische Mathematik* 1970; **15**(4):283–296, doi: 10.1007/BF02165119.
14. Boisserie JM. A new C^1 finite element: Full heptic. *International Journal for Numerical Methods in Engineering* 1989; **28**(3):667–677, doi:10.1002/nme.1620280313.
15. Mindlin RD. Micro-structure in linear elasticity. *Archive for Rational Mechanics and Analysis* 1964; **16**(1):51–78, doi:10.1007/BF00248490.
16. Mindlin RD, Eshel NN. On first strain-gradient theories in linear elasticity. *International Journal of Solids and Structures* 1968; **4**(1):109–124, doi:10.1016/0020-7683(68)90036-X.
17. Vardoulakis I. Linear micro-elasticity. *Degradations and instabilities in geomaterials*, Darve F, Vardoulakis I (eds.), CISM, Springer-Verlag, 2003; 107–149.
18. Papanicolopulos SA, Zervos A. Continua with microstructure: second-gradient theory. *European Journal of Environmental and Civil Engineering* 2010; **14**(8-9):1031–1050, doi:10.3166/ejece.14.1031-1050.